

NECESSARY CONDITIONS FOR THE EXISTENCE OF 3-DESIGNS OVER FINITE FIELDS WITH NONTRIVIAL AUTOMORPHISM GROUPS

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ABSTRACT. A q -design with parameters $t-(v, k, \lambda_t)_q$ is a pair $(\mathcal{V}, \mathcal{B})$ of the vector space $\mathcal{V} = \mathbb{F}_q^v$ and a collection \mathcal{B} of k -dimensional subspaces of \mathcal{V} , such that each t -dimensional subspace of \mathcal{V} is contained in precisely λ_t members of \mathcal{B} . In this paper we give new general necessary conditions on the existence of designs over finite fields with parameters $3-(v, k, \lambda_3)_q$ with a prescribed automorphism group. These necessary conditions are based on a tactical decomposition of such a design over a finite field and are given in the form of equations for the coefficients of tactical decomposition matrices. In particular, they represent necessary conditions on the existence of q -analogues of Steiner systems admitting a prescribed automorphism group.

1. INTRODUCTION

Definition 1.1. A q -design with parameters $t-(v, k, \lambda_t)_q$, or shorter a $t-(v, k, \lambda_t)_q$ design, with $v > k > 1, k \geq t \geq 1, \lambda_t \geq 1$, is a pair $(\mathcal{V}, \mathcal{B})$ of the vector space $\mathcal{V} = \mathbb{F}_q^v$ and a collection \mathcal{B} of k -dimensional subspaces of \mathcal{V} (called blocks), such that each t -dimensional subspace of \mathcal{V} is contained in precisely λ_t blocks.

If \mathcal{B} consists of all the k -dimensional subspaces of \mathcal{V} , the design $(\mathcal{V}, \mathcal{B})$ is called *trivial*. Throughout this paper we shall call the 1-spaces of \mathcal{V} *points* and often, when convenient, identify \mathcal{V} with its set of points.

q -designs recently attract considerable attention. They represent a generalisation of classical designs in terms of vector spaces, a so-called q -analogue, and were first introduced in the 1970's, see [6, 7, 8]. The recent application of these q -designs in error-correction in randomized network coding, made this topic more interesting than ever [10, 11].

In random network coding, information is transmitted through a network whose topology can vary. A classical example is a wireless

Date: September, 2015.

1991 Mathematics Subject Classification. 05B05.

Key words and phrases. q -design; tactical decomposition.

network where users come and go. For more details see [1, 13, 16, 19, 27]. It was showed in [16] that subspace codes are well-suited for transmission in networks. A *subspace code* is a set of k -dimensional vector subspaces of the vector space \mathbb{F}_q^v . This new insight led to many new interesting problems in coding theory, in Galois geometries and in design theory (for a general overview see [10]). E.g., it has been noted (see [1, 11]) that q -analogues of Steiner systems, briefly called *q -Steiner systems*, are optimal subspace codes. These q -Steiner systems are q -designs with $\lambda_t = 1$. The existence problem of q -Steiner systems is still open. It is known that a $1-(v, k, 1)_q$ Steiner system exists if and only if k divides v , and in this case it is called a spread. The first q -Steiner systems which are not spreads were constructed recently in [3]: using algorithms based on automorphism groups a $2-(13, 3, 1)_2$ design was computationally constructed. Currently, one of the most interesting open problems is the existence of the q -analogue of the Fano plane, a $2-(7, 3, 1)_q$ design (see [5, 11, 15, 22, 23, 26]).

Also for general q -designs the existence problem is mainly unsolved. Examples of designs over finite fields with $t \geq 2$ constructed so far are mostly q -designs with $t = 2$ [3, 4, 23, 24, 25, 26].

Therefore, an interesting research direction is the study of q -designs with $t > 2$. Only a few examples of non-trivial 3-designs over finite fields were constructed: a $3-(8, 4, 11)_2$ design and a $3-(8, 4, 20)_2$ design in [4], and a $3-(8, 4, 15)_2$ design in [2]. They were all constructed computationally with algorithms using their automorphism group.

One of the open problems posed in [10] is finding new necessary conditions on the existence of q -Steiner systems. In this paper we give new general necessary conditions on the existence of $3-(v, k, \lambda_3)_q$ designs with a prescribed automorphism group. These necessary conditions are based on a tactical decomposition of such a q -design and are given in the form of equations for the coefficients of tactical decomposition matrices. In [20] tactical decompositions of designs over finite fields with $t = 2$ were studied. It was shown there that coefficients of tactical decomposition matrices satisfy an equation system analogue to the one known for classical block designs (see Section 3). The main result is given in Theorem 4.4. Further in Section 4 we show that for $t = 3$, the system of equations for q -designs is not equivalent to the one for classical 3-designs. Crucial in the main theorem are the values Λ_{lrs} ; additional results about these Λ -values are presented in Section 5. Finally, in Section 6 we present some applications of the main theorem to known q -designs.

2. PRELIMINARIES

The number of r -dimensional subspaces of the vector space \mathbb{F}_q^v is

$$\begin{bmatrix} v \\ r \end{bmatrix}_q = \frac{(q^v - 1)(q^{v-1} - 1) \dots (q^{v-r+1} - 1)}{(q^r - 1)(q^{r-1} - 1) \dots (q - 1)}.$$

The number of r -dimensional subspaces of \mathcal{V} containing a fixed s -dimensional subspace, $s \leq r$, equals $\begin{bmatrix} v-s \\ r-s \end{bmatrix}_q$. For every two subspaces U and W of a vector space, *the dimension formula* is valid:

$$\dim \langle U, W \rangle = \dim U + \dim W - \dim (U \cap W).$$

If $(\mathcal{V}, \mathcal{B})$ is a t -(v, k, λ_t) $_q$ design, then it is also a q -design with parameters s -(v, k, λ_s) $_q$, $0 \leq s \leq t$, with

$$(1) \quad \lambda_s = \lambda_t \frac{\begin{bmatrix} v-s \\ t-s \end{bmatrix}_q}{\begin{bmatrix} k-s \\ t-s \end{bmatrix}_q}.$$

The number of blocks in \mathcal{B} equals

$$|\mathcal{B}| = \lambda_0 = \lambda_t \frac{\begin{bmatrix} v \\ t \end{bmatrix}_q}{\begin{bmatrix} k \\ t \end{bmatrix}_q}.$$

q -designs are closely related to classical designs, as they are the q -analogues of the classical designs. A t -(v, k, λ_t) *design* is a finite incidence structure $(\mathcal{P}, \mathcal{B})$, where \mathcal{P} is a set of v elements called *points*, and \mathcal{B} is a multiset of nonempty k -subsets of \mathcal{P} called *blocks* such that every set of t distinct points is contained in precisely λ_t blocks. Every 2 -(v, k, λ_2) $_q$ design gives rise to a classical design with parameters 2 - $\left(\begin{bmatrix} v \\ 1 \end{bmatrix}_q, \begin{bmatrix} k \\ 1 \end{bmatrix}_q, \lambda_2\right)$ by identifying the points of \mathcal{V} with the points of the design and each block in \mathcal{B} with the set of points it contains. The inverse statement is not valid. E.g. there are classical designs with parameters 2 -(15, 7, 3) which cannot be constructed from the unique 2 -(4, 3, 3) $_2$ design [21].

An *automorphism* of the q -design $(\mathcal{V}, \mathcal{B})$ is a map $g \in \text{PTL}(\mathcal{V})$ such that $\mathcal{B}^g = \mathcal{B}$. The set $\text{Aut}(\mathcal{V}, \mathcal{B})$ of all automorphisms of $(\mathcal{V}, \mathcal{B})$ is a subgroup of $\text{PTL}(\mathcal{V})$, called the *full automorphism group* of $(\mathcal{V}, \mathcal{B})$. We say that $(\mathcal{V}, \mathcal{B})$ admits the finite group G , or equivalently that G is

an automorphism group of $(\mathcal{V}, \mathcal{B})$, if there is a subgroup of $\text{Aut}(\mathcal{V}, \mathcal{B})$ isomorphic to G .

3. q -DESIGNS WITH A TACTICAL DECOMPOSITION

In this section we address the definition and known results concerning automorphism groups and tactical decompositions of q -designs. The idea of considering tactical decompositions of classical block designs was first introduced by Dembowski [9]. Tactical decomposition has been crucial for the construction of many classical 2-designs [14, 21]. In [20] tactical decompositions of q -designs with $t = 2$ were studied.

Definition 3.1. *Let $(\mathcal{V}, \mathcal{B})$ be a q -design. A decomposition of $(\mathcal{V}, \mathcal{B})$ consists of two partitions*

$$\mathcal{V} = \mathcal{V}_1 \sqcup \cdots \sqcup \mathcal{V}_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n.$$

We say that a decomposition is tactical if there exist nonnegative integers ρ_{ij} , κ_{ij} , $i = 1, \dots, m$, $j = 1, \dots, n$, such that

- (1) *every point of \mathcal{V}_i is contained in ρ_{ij} blocks of \mathcal{B}_j ,*
- (2) *each block of \mathcal{B}_j contains κ_{ij} points of \mathcal{V}_i .*

The matrices $\mathcal{R} = [\rho_{ij}]$ and $\mathcal{K} = [\kappa_{ij}]$ are called the tactical decomposition matrices.

There are two trivial examples of a tactical decomposition of a q -design. The first example is obtained by putting $n = m = 1$, and the second by partitioning sets \mathcal{V} and \mathcal{B} into singletons. A nontrivial tactical decomposition can be obtained by the action of an automorphism group $G \leq \text{Aut}(\mathcal{V}, \mathcal{B})$ on a design.

Theorem 3.2. *Let G be an automorphism group of a design $(\mathcal{V}, \mathcal{B})$ over a finite field. Then the orbits of the set of points \mathcal{V} and the orbits of the set of blocks \mathcal{B} form a tactical decomposition.*

Proof. Let \mathcal{V}_i be a point orbit and \mathcal{B}_j be an orbit of \mathcal{B} under the action of G . The statement follows immediately from the observation that $P \in \mathcal{V}_i$ is contained in $B \in \mathcal{B}_j$ if and only if $P^g \in \mathcal{V}_i$ is contained in $B^g \in \mathcal{B}_j$ for any $g \in G$. \square

A tactical decomposition that arises from a group action as in Theorem 3.2 is called *group-induced*. If the group G is specified, we call it *G -induced*.

The following result is valid for all designs over finite fields.

Lemma 3.3 ([20, Section 2]). *Let $(\mathcal{V}, \mathcal{B})$ be a design with parameters t -(v, k, λ_t) $_q$ that admits a tactical decomposition*

$$\mathcal{V} = \mathcal{V}_1 \sqcup \cdots \sqcup \mathcal{V}_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n,$$

with tactical decomposition matrices $\mathcal{R} = [\rho_{ij}]$ and $\mathcal{K} = [\kappa_{ij}]$. Then,

$$\sum_{i=1}^m \kappa_{ij} = \begin{bmatrix} k \\ 1 \end{bmatrix}_q, \quad \sum_{j=1}^n \rho_{ij} = \lambda_1,$$

and for all $i = 1, \dots, m$ and $j = 1, \dots, n$,

$$|\mathcal{V}_i| \cdot \rho_{ij} = |\mathcal{B}_j| \cdot \kappa_{ij}.$$

It was shown in [20] that coefficients of a tactical decomposition matrix of a q -design with $t = 2$ satisfy an equation system analogous to the one known for classical block designs with $t = 2$.

Theorem 3.4. [20] *Assume $(\mathcal{V}, \mathcal{B})$ is a $2-(v, k, \lambda_2)_q$ design over finite field or a classical $2-(v, k, \lambda_2)$ design with a tactical decomposition*

$$\mathcal{V} = \mathcal{V}_1 \sqcup \dots \sqcup \mathcal{V}_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \dots \sqcup \mathcal{B}_n.$$

Let $[\rho_{ij}]$ and $[\kappa_{ij}]$ be the associated tactical decomposition matrices. Then

$$(2) \quad \sum_{j=1}^n \rho_{lj} \kappa_{rj} = \begin{cases} \lambda_1 + \lambda_2 \cdot (|\mathcal{V}_r| - 1) & l = r \\ \lambda_2 \cdot |\mathcal{V}_r| & l \neq r. \end{cases}$$

Note that the right-hand side of (2) only contains parameters of the design that can easily be computed.

4. 3-DESIGNS OVER FINITE FIELDS WITH NONTRIVIAL AUTOMORPHISM GROUPS

We now investigate designs over finite fields with $t = 3$ having a nontrivial tactical decomposition. We introduce the following notation.

Definition 4.1. *Let $\mathcal{V} = \mathcal{V}_1 \sqcup \dots \sqcup \mathcal{V}_m$ be a partition of the vector space \mathcal{V} . For a point $P \in \mathcal{V}_l$, we define the parameters*

$$\Lambda_{rs}(P) = |\{(R, S) \in \mathcal{V}_r \times \mathcal{V}_s : \dim \langle P, R, S \rangle = 2\}|.$$

Lemma 4.2. *Let $(\mathcal{V}, \mathcal{B})$ be a $3-(v, k, \lambda_3)_q$ design that admits a tactical decomposition*

$$\mathcal{V} = \mathcal{V}_1 \sqcup \dots \sqcup \mathcal{V}_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \dots \sqcup \mathcal{B}_n,$$

with tactical decomposition matrices $\mathcal{R} = [\rho_{ij}]$ and $\mathcal{K} = [\kappa_{ij}]$. Let P be a point in \mathcal{V}_l . Then,

$$\sum_{j=1}^n \rho_{lj} \kappa_{rj} \kappa_{sj} = \begin{cases} \lambda_1 + \Lambda_l(P) \cdot \lambda_2 + (|\mathcal{V}_l|^2 - \Lambda_l(P) - 1) \cdot \lambda_3 & l = r = s \\ \Lambda_{rs}(P) \cdot \lambda_2 + (|\mathcal{V}_r| \cdot |\mathcal{V}_s| - \Lambda_{rs}(P)) \cdot \lambda_3 & \text{otherwise.} \end{cases}$$

Proof. Double-counting of the set of triples

$$\{(R, S, B) \in \mathcal{V}_r \times \mathcal{V}_s \times \mathcal{B} : P, R, S \leq B\}$$

yields

$$(3) \quad \sum_{j=1}^n \rho_{lj} \kappa_{rj} \kappa_{sj} = \sum_{R \in \mathcal{V}_r} \sum_{S \in \mathcal{V}_s} |\mathcal{I}_P \cap \mathcal{I}_R \cap \mathcal{I}_S|,$$

with \mathcal{I}_Q the subset of blocks of \mathcal{B} that contain the point Q , for any point Q . Now, consider $R \in \mathcal{V}_r$ and $S \in \mathcal{V}_s$. It is immediate that

$$\mathcal{I}_P \cap \mathcal{I}_R \cap \mathcal{I}_S = \{B \in \mathcal{B} : \langle P, R, S \rangle \leq B\}.$$

It is clear that $1 \leq \dim \langle P, R, S \rangle \leq 3$ and so

$$|\mathcal{I}_P \cap \mathcal{I}_R \cap \mathcal{I}_S| = \lambda_{\dim \langle P, R, S \rangle}.$$

Hence, in order to find an expression for (3) it is sufficient to count the number of pairs $(R, S) \in \mathcal{V}_r \times \mathcal{V}_s$ such that $\dim \langle P, R, S \rangle = i$, for $i = 1, 2, 3$. It is clear that $\dim \langle P, R, S \rangle = 1$ if and only if $P = R = S$ and thus $l = r = s$. Consequently,

$$\begin{aligned} & \sum_{R \in \mathcal{V}_r} \sum_{S \in \mathcal{V}_s} |\mathcal{I}_P \cap \mathcal{I}_R \cap \mathcal{I}_S| \\ &= \begin{cases} \lambda_1 + \Lambda_{ll}(P) \cdot \lambda_2 + (|\mathcal{V}_l|^2 - \Lambda_{ll}(P) - 1) \cdot \lambda_3, & l = r = s \\ \Lambda_{rs}(P) \cdot \lambda_2 + (|\mathcal{V}_r| \cdot |\mathcal{V}_s| - \Lambda_{rs}(P)) \cdot \lambda_3, & \text{otherwise,} \end{cases} \end{aligned}$$

from which the result follows, using (3). \square

Corollary 4.3. *Let $(\mathcal{V}, \mathcal{B})$ be a $3-(v, k, \lambda_3)_q$ design that admits a tactical decomposition*

$$\mathcal{V} = \mathcal{V}_1 \sqcup \cdots \sqcup \mathcal{V}_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n.$$

Consider $l, r, s \in \{1, \dots, m\}$. The values $\Lambda_{rs}(P)$ are independent of the choice of $P \in \mathcal{V}_l$.

Proof. Let $\mathcal{R} = [\rho_{ij}]$ and $\mathcal{K} = [\kappa_{ij}]$ be the tactical decomposition matrices of this tactical decomposition. By Lemma 4.2 we know that

$$\sum_{j=1}^n \rho_{lj} \kappa_{rj} \kappa_{sj} = \begin{cases} \Lambda_{ll}(P) \cdot (\lambda_2 - \lambda_3) + \lambda_1 + (|\mathcal{V}_l|^2 - 1) \cdot \lambda_3, & l = r = s \\ \Lambda_{rs}(P) \cdot (\lambda_2 - \lambda_3) + |\mathcal{V}_r| \cdot |\mathcal{V}_s| \cdot \lambda_3, & \text{otherwise.} \end{cases}$$

for a point $P \in \mathcal{V}_l$. The left-hand side in this equation is clearly independent of the choice of P , hence also the right-hand side is independent of the choice of P . As $\lambda_1, \lambda_2, \lambda_3, |\mathcal{V}_r|, |\mathcal{V}_s|$ are obviously P -independent and since $\lambda_2 - \lambda_3 \neq 0$, necessarily also $\Lambda_{rs}(P)$ is independent of the choice of $P \in \mathcal{V}_l$. \square

Following the result of Corollary 4.3, we can define Λ_{lrs} as $\Lambda_{rs}(P)$ for a point $P \in \mathcal{V}_l$. Using this notation, we state the main theorem of this section.

Theorem 4.4. *Let $(\mathcal{V}, \mathcal{B})$ be a design over finite field with parameters $3-(v, k, \lambda_3)_q$ that admits a tactical decomposition*

$$\mathcal{V} = \mathcal{V}_1 \sqcup \cdots \sqcup \mathcal{V}_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n,$$

with tactical decomposition matrices $\mathcal{R} = [\rho_{ij}]$ and $\mathcal{K} = [\kappa_{ij}]$. Then,

$$\sum_{j=1}^n \rho_{lj} \kappa_{rj} \kappa_{sj} = \begin{cases} \lambda_1 + \Lambda_{lll} \cdot \lambda_2 + (|\mathcal{V}_l|^2 - \Lambda_{lll} - 1) \cdot \lambda_3 & l = r = s \\ \Lambda_{lrs} \cdot \lambda_2 + (|\mathcal{V}_r| \cdot |\mathcal{V}_s| - \Lambda_{lrs}) \cdot \lambda_3 & \text{otherwise.} \end{cases}$$

Note that a q -design with $t = 3$ is also a q -design with $t = 2$. Hence, Lemma 3.3 and Lemma 3.4 also present necessary conditions for the existence of q -designs with $t = 3$, with a given tactical decomposition.

These parameters Λ_{lrs} are not present in the discussion of classical 3-designs as in [17, 18], and form the main difference between the designs over finite fields and the classical designs with $t = 3$. In order to compare we state here the analogue of Theorem 4.4 for classical designs with $t = 3$.

Theorem 4.5. [17, 18] *Let $(\mathcal{V}, \mathcal{B})$ be a $3-(v, k, \lambda_3)$ design that admits a tactical decomposition*

$$\mathcal{V} = \mathcal{V}_1 \sqcup \cdots \sqcup \mathcal{V}_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n,$$

with tactical decomposition matrices $\mathcal{R} = [\rho_{ij}]$ and $\mathcal{K} = [\kappa_{ij}]$. Then,

$$\begin{aligned} & \sum_{j=1}^n \rho_{lj} \kappa_{rj} \kappa_{sj} \\ &= \begin{cases} \lambda_1 + 3(|\mathcal{V}_l| - 1) \cdot \lambda_2 + (|\mathcal{V}_l| - 1) \cdot (|\mathcal{V}_l| - 2) \cdot \lambda_3 & l = r = s \\ \lambda_2 \cdot |\mathcal{V}_r| + \lambda_3 \cdot |\mathcal{V}_r| \cdot (|\mathcal{V}_r| - 1) & l \neq r = s \\ |\mathcal{V}_r| \cdot |\mathcal{V}_s| \cdot \lambda_3 + (\lambda_2 - \lambda_3) \cdot (\delta_{lr} \cdot |\mathcal{V}_s| + \delta_{ls} \cdot |\mathcal{V}_r|) & \text{otherwise.} \end{cases} \end{aligned}$$

5. SOME RESULTS REGARDING Λ_{lrs}

In this section we have a look at the parameter Λ_{lrs} which we introduced in Definition 4.1. In order to use Theorem 4.4 when computationally constructing a q -design, one needs to know these values Λ_{lrs} for the given tactical decomposition. First we present three general results on these values Λ_{lrs} and then we look at a specific case.

Theorem 5.1. *Let $(\mathcal{V}, \mathcal{B})$ be a $3\text{--}(v, k, \lambda_3)_q$ design that admits a tactical decomposition*

$$\mathcal{V} = \mathcal{V}_1 \sqcup \cdots \sqcup \mathcal{V}_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n.$$

Then $\Lambda_{lrs} = \Lambda_{lsr}$ and $|\mathcal{V}_l| \cdot \Lambda_{lrs} = |\mathcal{V}_r| \cdot \Lambda_{rls}$.

Proof. It follows directly from Definition 4.1 that $\Lambda_{rs}(P) = \Lambda_{sr}(P)$ for any point $P \in \mathcal{V}_l$, and hence that $\Lambda_{lrs} = \Lambda_{lsr}$.

The equality $|\mathcal{V}_l| \Lambda_{lrs} = |\mathcal{V}_r| \Lambda_{rls}$ surely holds if $l = r$, so we can assume $l \neq r$. Let $\mathcal{R} = [\rho_{ij}]$ and $\mathcal{K} = [\kappa_{ij}]$ be the tactical decomposition matrices of the given tactical decomposition. Using Lemma 3.3 we find that

$$\sum_{j=1}^n \rho_{lj} \kappa_{rj} \kappa_{sj} = \sum_{j=1}^n \frac{|\mathcal{B}_j|}{|\mathcal{V}_l|} \kappa_{lj} \kappa_{rj} \kappa_{sj} = \sum_{j=1}^n \frac{|\mathcal{V}_r|}{|\mathcal{V}_l|} \rho_{rj} \kappa_{lj} \kappa_{sj},$$

hence

$$|\mathcal{V}_l| \sum_{j=1}^n \rho_{lj} \kappa_{rj} \kappa_{sj} = |\mathcal{V}_r| \sum_{j=1}^n \rho_{rj} \kappa_{lj} \kappa_{sj}.$$

Applying Theorem 4.4 we find

$$\begin{aligned} |\mathcal{V}_l| (\Lambda_{lrs}(\lambda_2 - \lambda_3) + \lambda_3 |\mathcal{V}_r| \cdot |\mathcal{V}_s|) &= |\mathcal{V}_r| (\Lambda_{rls}(\lambda_2 - \lambda_3) + \lambda_3 |\mathcal{V}_l| \cdot |\mathcal{V}_s|) \\ \Leftrightarrow |\mathcal{V}_l| \Lambda_{lrs}(\lambda_2 - \lambda_3) &= |\mathcal{V}_r| \Lambda_{rls}(\lambda_2 - \lambda_3), \end{aligned}$$

whence the equality $|\mathcal{V}_l| \Lambda_{lrs} = |\mathcal{V}_r| \Lambda_{rls}$ since $\lambda_2 - \lambda_3 \neq 0$. \square

Theorem 5.2. *Let $(\mathcal{V}, \mathcal{B})$ be a $3\text{--}(v, k, \lambda_3)_q$ design that admits a tactical decomposition*

$$\mathcal{V} = \mathcal{V}_1 \sqcup \cdots \sqcup \mathcal{V}_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n.$$

Then

$$\sum_{s=1}^m \Lambda_{lrs} = \begin{cases} |\mathcal{V}_l|(q+1) + \frac{q^v - q^2}{q-1} - 1 & l = r \\ |\mathcal{V}_r|(q+1) & l \neq r. \end{cases}$$

Proof. Let P be a point of \mathcal{V}_l . We count the set

$$\{(R, S) \in \mathcal{V}_r \times \mathcal{V} : \dim \langle P, R, S \rangle = 2\}$$

in two ways. On the one hand,

$$\begin{aligned} &\{(R, S) \in \mathcal{V}_r \times \mathcal{V} : \dim \langle P, R, S \rangle = 2\} \\ &= \bigsqcup_{s=1}^m \{(R, S) \in \mathcal{V}_r \times \mathcal{V}_s : \dim \langle P, R, S \rangle = 2\}, \end{aligned}$$

so the size of this set equals $\sum_{s=1}^m \Lambda_{lrs}$ by Definition 4.1 and Corollary 4.3.

On the other hand, if $l \neq r$, then for any point $R \in \mathcal{V}_r$ we find $q + 1$ different 1-spaces in the 2-dimensional space $\langle P, R \rangle$, so $q + 1$ choices for the point S , therefore the size of this set equals

$$|\mathcal{V}_r|(q + 1).$$

If $l = r$, then for any point in $R \in \mathcal{V}_r \setminus \{P\}$ we find $q + 1$ different 1-spaces in the 2-dimensional space $\langle P, R \rangle$, so $q + 1$ choices for the point S . For the point $R = P$, any point $S \in \mathcal{V} \setminus \{P\}$ determines a 2-dimensional space $\langle P, R, S \rangle = \langle P, S \rangle$. Hence, the size of this set for $l = r$ equals

$$(|\mathcal{V}_l| - 1)(q + 1) + \left(\frac{q^v - 1}{q - 1} - 1 \right) = |\mathcal{V}_l|(q + 1) + \frac{q^v - q^2}{q - 1} - 1.$$

This concludes the proof. \square

Recall that the 1-dimensional subspaces of \mathcal{V} are called points. From now on, we call the 2-dimensional subspaces of \mathcal{V} *lines*. The set of lines of \mathcal{V} will often be denoted by \mathcal{L} . Note that the pair $(\mathcal{V}, \mathcal{L})$ is a trivial $2-(v, 2, 1)_q$ design. Every group $G \leq \text{PTL}(\mathcal{V})$ induces a tactical decomposition on $(\mathcal{V}, \mathcal{L})$. This obtained tactical decomposition is closely related to parameter Λ_{rls} .

Theorem 5.3. *Let $(\mathcal{V}, \mathcal{B})$ be a $3-(v, k, \lambda_3)_q$ design that admits a G -induced tactical decomposition*

$$\mathcal{V} = \mathcal{V}_1 \sqcup \cdots \sqcup \mathcal{V}_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n.$$

Let \mathcal{L} be the set of lines of \mathcal{V} . We consider the G -induced tactical decomposition of the trivial design $(\mathcal{V}, \mathcal{L})$:

$$\mathcal{V} = \mathcal{V}_1 \sqcup \cdots \sqcup \mathcal{V}_m, \quad \mathcal{L} = \mathcal{L}_1 \sqcup \cdots \sqcup \mathcal{L}_\omega,$$

with associated tactical decomposition matrices $[\rho_{ij}^{\mathcal{L}}]$ and $[\kappa_{ij}^{\mathcal{L}}]$. Then

$$\Lambda_{rls} = \begin{cases} \sum_{j=1}^{\omega} \rho_{lj}^{\mathcal{L}} \kappa_{rj}^{\mathcal{L}} \kappa_{sj}^{\mathcal{L}} - \begin{bmatrix} v-1 \\ 1 \end{bmatrix}_q & l = r = s \\ \sum_{j=1}^{\omega} \rho_{lj}^{\mathcal{L}} \kappa_{rj}^{\mathcal{L}} \kappa_{sj}^{\mathcal{L}} & \text{otherwise.} \end{cases}$$

Proof. Let P be a point of \mathcal{V}_l . From the definition of the coefficients $\rho_{ij}^{\mathcal{L}}$ and $\kappa_{ij}^{\mathcal{L}}$ it follows that each point of \mathcal{V}_i lies on $\rho_{ij}^{\mathcal{L}}$ lines of \mathcal{L}_j , and each line of \mathcal{L}_j contains $\kappa_{ij}^{\mathcal{L}}$ points of \mathcal{V}_i . Since $(\mathcal{V}, \mathcal{L})$ is a q -Steiner system with $t = 2$ we know that

$$\begin{aligned} \Lambda_{lrs} &= |\{(R, S) \in \mathcal{V}_r \times \mathcal{V}_s : \dim \langle P, R, S \rangle = 2\}| \\ &= |\{(R, S, \ell) \in \mathcal{V}_r \times \mathcal{V}_s \times \mathcal{L} : \langle P, R, S \rangle = \ell\}|. \end{aligned}$$

Counting the second set yields

$$\Lambda_{rls} = \begin{cases} \sum_{j=1}^{\omega} \rho_{lj}^{\mathcal{L}} \kappa_{rj}^{\mathcal{L}} \kappa_{sj}^{\mathcal{L}} - \begin{bmatrix} v-1 \\ 1 \end{bmatrix}_q & l = r = s \\ \sum_{j=1}^{\omega} \rho_{lj}^{\mathcal{L}} \kappa_{rj}^{\mathcal{L}} \kappa_{sj}^{\mathcal{L}} & \text{otherwise.} \end{cases} \quad \square$$

Remark 5.4. Recall that only a few examples of non-trivial 3-designs over finite fields are known. These non-trivial examples were obtained using an automorphism group G acting transitively on the set of points of the a $3-(v, k, \lambda_3)_q$ design $(\mathcal{V}, \mathcal{B})$. In this case (if the group is acting transitively on the set of points of $(\mathcal{V}, \mathcal{B})$) there only one Λ parameter, namely Λ_{111} . As a corollary of Lemma 5.3 or directly using Definition 4.1 one can prove that

$$\Lambda_{111} = q(q+2) \begin{bmatrix} v-1 \\ 1 \end{bmatrix}_q.$$

Unfortunately, if G acts transitively on the points of \mathcal{V} the results of Lemma 3.4 and Theorem 4.4 cannot be used. For in this case, by Lemma 3.3 we know that $\kappa_{1j} = \begin{bmatrix} k \\ 1 \end{bmatrix}_q$ and hence the results of Lemma 3.4 and Theorem 4.4 reduce to $\lambda_1 = \sum_{j=1}^n \rho_{1j}$ by (1), a result we already know from Lemma 3.3.

The next theorem and the subsequent remark deal with G -induced tactical decompositions, with $|G| = p$ prime, having a fixed point. Note that such a fixed point (orbit of size one) is guaranteed to exist if p is not a divisor of the number of points.

Theorem 5.5. Let $(\mathcal{V}, \mathcal{B})$ be a $3-(v, k, \lambda_3)_q$ design and let

$$\mathcal{V} = \mathcal{V}_1 \sqcup \cdots \sqcup \mathcal{V}_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n,$$

be a G -induced tactical decomposition, $G \leq \text{Aut}(\mathcal{V}, \mathcal{B})$, with $|G| = p$ prime. If \mathcal{V}_l is an orbit of size one, then $\Lambda_{lrs} \in \{0, 1, p, p^2\}$ for $r, s = 1, \dots, m$.

Proof. Since $|G|$ is prime the orbits \mathcal{V}_i have size 1 or p . Let P be the unique point in \mathcal{V}_l . A line ℓ through P is either fixed by G or else ℓ^G is an orbit of p different lines through P . An orbit \mathcal{V}_i of size 1, $i \neq l$, is necessarily contained in a line through P that is fixed by G ; an orbit \mathcal{V}_i of size p is contained in a line through P that is fixed or has one point in common with each line of an orbit ℓ^G of p lines.

If $l = r = s$, then $\Lambda_{lrs} = 0$. If $l = r \neq s$, then $\Lambda_{lrs} = |\mathcal{V}_s|$ (analogously if $l = s \neq r$). Now, we assume that $r \neq l \neq s$. If \mathcal{V}_r is an orbit of size

one, equal to $\{R\}$, then the line $\langle P, R \rangle$ is fixed, and Λ_{lrs} equals 0 or $|\mathcal{V}_s|$. If \mathcal{V}_s is an orbit of size one, the situation is equivalent. If both \mathcal{V}_r and \mathcal{V}_s are orbits of size p , then Λ_{lrs} equals p^2 (in case \mathcal{V}_r and \mathcal{V}_s are on the same fixed line), p (in case \mathcal{V}_r and \mathcal{V}_s have one point in common with each line of the same line orbit ℓ^G) or 0 (else). \square

Remark 5.6. *Given a G -induced tactical decomposition of a q -design $(\mathcal{V}, \mathcal{B})$ with $t = 3$ and $|G| = p$ prime, we want to compute the Λ -values related to at least one orbit of size 1. Let $\mathcal{V} = \mathcal{V}_1 \sqcup \cdots \sqcup \mathcal{V}_m$ be the point set decomposition and let $\mathcal{V}_l = \{P\}$ be an orbit of size 1. A line ℓ through P is either fixed by G or is contained in a line orbit of size p . If ℓ is fixed, it contains only entire orbits of points. If ℓ is not fixed, then no two points on ℓ belong to the same point orbit. Hence, we can define a partition Ω of the set $\{\mathcal{V}_1, \dots, \mathcal{V}_{l-1}, \mathcal{V}_{l+1}, \dots, \mathcal{V}_m\}$ in the following way: two orbits are in the same partition class if and only if they are on the same fixed line through P or if they have one point in common with each line of the same line orbit of size p through P . We find*

$$\Omega = \Omega_1 \sqcup \cdots \sqcup \Omega_a \sqcup \Omega_{a+1} \sqcup \cdots \sqcup \Omega_b,$$

with each $\Omega_i = \{\mathcal{V}_{j_1}, \dots, \mathcal{V}_{j_{k_i}}\}$. Let $\Omega_1, \dots, \Omega_a$ be the partition classes that correspond to fixed lines and let $\Omega_{a+1}, \dots, \Omega_b$ be the partition classes that correspond to line orbits through P of size p .

Let $r \neq l \neq s$. Following the arguments in the proof of Theorem 5.5 the value Λ_{lrs} equals 0 if \mathcal{V}_r and \mathcal{V}_s do not belong to the same partition class Ω_i . If they do belong to the same partition class Ω_i , then $\Lambda_{lrs} = |\mathcal{V}_r| \cdot |\mathcal{V}_s|$ if $1 \leq i \leq a$ and $\Lambda_{lrs} = |\mathcal{V}_r| = |\mathcal{V}_s| = p$ if $a < i \leq b$. To summarize,

$$\Lambda_{lrs} = \begin{cases} |\mathcal{V}_r| \cdot |\mathcal{V}_s| & \mathcal{V}_r, \mathcal{V}_s \in \Omega_i, i \leq a, \\ p & \mathcal{V}_r, \mathcal{V}_s \in \Omega_i, a < i \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

The final theorem in this section gives a bound on the value of Λ_{lrs} given a tactical decomposition induced by a group of prime order. Considering Remark 5.6 this is of interest when the three orbits involved are not fixed points.

Theorem 5.7. *Let $(\mathcal{V}, \mathcal{B})$ be a 3 -(v, k, λ_3) $_q$ design and let*

$$\mathcal{V} = \mathcal{V}_1 \sqcup \cdots \sqcup \mathcal{V}_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n,$$

be a G -induced tactical decomposition, $G \leq \text{Aut}(\mathcal{V}, \mathcal{B})$, with $|G| = p$ prime. If $\Lambda_{lrs} \neq p^2$, then $\Lambda_{lrs} \leq p\sqrt{p - \frac{3}{4}} + \frac{p}{2}$, for $l, r, s = 1, \dots, m$.

Proof. Let \mathcal{V}_l , \mathcal{V}_r and \mathcal{V}_s be three point orbits. If $1 \in \{|\mathcal{V}_l|, |\mathcal{V}_r|, |\mathcal{V}_s|\}$, the result follows from Theorem 5.5. So, we assume $|\mathcal{V}_l| = |\mathcal{V}_r| = |\mathcal{V}_s| = p$. Let P be a point of \mathcal{V}_l .

We first show that an orbit \mathcal{V}_j , $j \neq l$ and with $|\mathcal{V}_j| = p$, has either p or else at most $K_p = \sqrt{p - \frac{3}{4}} + \frac{1}{2}$ points in common with a line through P . Let ℓ be a line through P having at least one point Q in common with \mathcal{V}_j . If $n_\ell = |\ell \cap \mathcal{V}_j| < p$, then $\mathcal{V}_j \not\subseteq \ell$, hence the line ℓ is not fixed, so ℓ^G is a set of p different lines. Now, let $\{g_1, \dots, g_{n_\ell}\}$ be the elements of G that map the points of $\ell \cap \mathcal{V}_j$ onto Q . The lines $\ell^{g_1}, \dots, \ell^{g_{n_\ell}}$ are n_ℓ different lines through Q , each containing n_ℓ distinct points of \mathcal{V}_j (including Q). Hence, we find at least $n_\ell(n_\ell - 1) + 1$ points of \mathcal{V}_j . As $|\mathcal{V}_j| = p$, necessarily $n_\ell^2 - n_\ell + 1 \leq p$. It follows that $n_\ell \leq \sqrt{p - \frac{3}{4}} + \frac{1}{2}$.

Now let ℓ_1, \dots, ℓ_c be the lines through P containing at least one point of \mathcal{V}_r and at least one point of \mathcal{V}_s . Denote $|\ell_i \cap \mathcal{V}_r|$ and $|\ell_i \cap \mathcal{V}_s|$ by n_i and n'_i respectively, $i = 1, \dots, c$. Then $\Lambda_{lrs} = \sum_{i=1}^c n_i n'_i$. Moreover, $\sum_{i=1}^c n_i \leq p$ and $\sum_{i=1}^c n'_i \leq p$. If there is a line through P containing p points of \mathcal{V}_r , then this line is fixed (necessarily $c \in \{0, 1\}$). If \mathcal{V}_s is also contained in this line, then $\Lambda_{lrs} = p^2$ and if it is not, then $\Lambda_{lrs} = 0$. So, now we can assume that $1 \leq n_i, n'_i \leq K_p$. By the Cauchy-Schwarz inequality

$$\sum_{i=1}^c n_i n'_i \leq \sqrt{\left(\sum_{i=1}^c n_i^2\right) \left(\sum_{i=1}^c n_i'^2\right)}.$$

Since $1 \leq n_i, n'_i \leq K_p$, it is immediate that

$$\sum_{i=1}^c n_i^2 \leq \frac{p}{K_p} K_p^2 = p K_p \text{ and analogously } \sum_{i=1}^c n_i'^2 \leq p K_p.$$

So,

$$\sum_{i=1}^c n_i n'_i \leq \sqrt{(p K_p)(p K_p)} = p K_p,$$

which proves the inequality. \square

6. APPLICATION TO KNOWN DESIGNS

In this final section we will discuss the application of the results in Section 4 to some known 3-designs. First we look at a design $(\mathcal{V}, \mathcal{B})$ with parameters $3-(4, 3, 1)_2$, where $\mathcal{V} = \mathbb{F}_2^4$. One can see directly that this is the design of all 3-spaces in \mathcal{V} , but as proof of concept we construct it using a tactical decomposition based on a prescribed automorphism group. We consider the group $G \leq \text{P}\Gamma\text{L}(\mathcal{V})$ generated

by

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

The group G is a cyclic group of order 3; its action on the points of \mathcal{V} yields five orbits:

$$\mathcal{V} = \mathcal{V}_1 \sqcup \cdots \sqcup \mathcal{V}_5,$$

with orbit representatives $\langle [1, 0, 0, 0] \rangle$, $\langle [1, 0, 1, 0] \rangle$, $\langle [1, 0, 1, 1] \rangle$, $\langle [1, 1, 0, 0] \rangle$ and $\langle [0, 1, 0, 0] \rangle$. Each of these five orbits has size 3. Now we assume that G is an automorphism group of the design $(\mathcal{V}, \mathcal{B})$. We do not know the orbits of \mathcal{B} under the action of G as we do not know which 3-spaces of \mathcal{V} are blocks of $(\mathcal{V}, \mathcal{B})$. However, all orbits of \mathcal{B} must have size 3 since no 3-space of \mathcal{V} can be fixed by G . Indeed, a fixed 3-space contains only full point orbits. Since each point orbit is of size 3, and a 3-space contains 7 points, G cannot fix any 3-space of \mathcal{V} . So, and since $|\mathcal{B}| = 15$ we can write

$$\mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_5,$$

with $\mathcal{B}_1, \dots, \mathcal{B}_5$ the orbits of \mathcal{B} under the action of G .

We consider the corresponding tactical decomposition matrices $[\rho_{ij}]$ and $[\kappa_{ij}]$. Note that by Lemma 3.3 we have $\rho_{ij} = \kappa_{ij}$ for all i, j , since all point orbits and all block orbits have size 3. First we need to calculate the values Λ_{lrs} for $l, r, s \in \{1, \dots, 5\}$. An easy calculation yields

$$\Lambda_{lrs} = \begin{cases} 8 & l = r = s \\ 1 & l \neq r \neq s \neq l \\ 3 & \text{else.} \end{cases}$$

Now we apply Lemma 3.3, Theorem 3.4 and Theorem 4.4. Note that $\lambda_3 = 1$, $\lambda_2 = 3$ and $\lambda_1 = 7$. We find

$$(4) \quad \sum_{j=1}^5 \rho_{ij} = 7, \quad \sum_{i=1}^5 \rho_{ij} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_2 = 7,$$

$$(5) \quad \sum_{j=1}^5 \rho_{lj} \rho_{rj} = \begin{cases} 13 & l = r \\ 9 & l \neq r, \end{cases}$$

$$(6) \quad \sum_{j=1}^5 \rho_{lj} \rho_{rj} \rho_{sj} = \begin{cases} 31 & l = r = s \\ 11 & l \neq r \neq s \neq l \\ 15 & \text{else.} \end{cases}$$

By looking at the relations (4), (5), (6) for a row (row sum, row sum of squares, row sum of third powers) it follows directly that a row of the matrix $[\rho_{ij}]$ must be a permutation of

$$\{3, 1, 1, 1, 1\}.$$

By using relation (5) it can be seen that no two 3's can be in the same column. Hence, the matrix $[\rho_{ij}]$ should equal

$$\begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix}$$

up to a rearrangement of rows and columns, and indeed this matrix satisfies all of the above relations. We find the design $(\mathcal{V}, \mathcal{B})$ consisting of all 3-spaces in $\mathcal{V} = \mathbb{F}_2^4$.

Now we look at the $3 - (8, 4, \lambda)_2$ designs $(\mathcal{V}, \mathcal{B})$ that were studied in [4]. Here $\mathcal{V} = \mathbb{F}_2^8$ and \mathcal{B} is a set of 4-spaces. These q -designs admit (by construction) the normaliser of a Singer cycle as an automorphism group; this automorphism group G is generated by the elements

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The group G acts transitively on the point set of \mathcal{V} . In this case the comments of Remark 5.4 apply and we need only to consider the result from Lemma 3.3. The group G has 109 orbits on the 4-spaces of \mathbb{F}_2^8 and we know that a $3 - (8, 4, \lambda)_2$ design which admits G as its automorphism group, is the union of some of these orbits, $\mathcal{B}_1, \dots, \mathcal{B}_n$.

Since $|\mathcal{V}_1| = |\mathcal{V}| = 255$ and $\kappa_{1j} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_2 = 15$ we know that $\rho_{1j} = \frac{|\mathcal{B}_j|}{17}$.

Considering the orbits of the 4-spaces under the action of G , we find that there are 92 orbits having size $120 \cdot 17$, ten orbits having size $60 \cdot 17$, five orbits having size $30 \cdot 17$, one orbit having size $20 \cdot 17$ and one orbit

having size 17. By (1) and Lemma 3.3 we obtain that

$$\sum_{j=1}^n \rho_{1j} = \lambda_1 = 127 \cdot 3 \cdot \lambda \equiv \lambda \pmod{10}.$$

Except possibly one, all values ρ_{ij} equal 0 modulo 10. Hence, λ must equal 1, 10, 11, 20, 21, 30 or 31. Note that $\lambda = 31$ corresponds to a trivial design and that 20, 21 and 30 can only occur as complements of 11, 10 and 1 respectively. In [4] all values from 1 up to 30 were tested for λ , but using tactical decomposition arguments that search could have been restricted to three values for λ . This shows the usefulness of this technique.

Acknowledgment: The research of the first author is supported by the BOF-UGent (Special Research Fund of Ghent University). The research of the second author has been supported by the Croatian Science Foundation under the project 7766.

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